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Mixed convection over a cooled horizontal plate: non-uniqueness and numerical instabilities of the boundary-layer equations

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The boundary-layer flow over a cooled horizontal plate is considered. It is shown that the real part of the spectrum of the evolution operator of the linearized equations is not bounded uniformly from above which explains the difficulties encounterd by a numerical solution. Furthermore it is shown that near the leading edge an asymptotic expansion of the solution is not unique. A one-parametric family of asymptotic expansions of solutions can be constructed.

1. Introduction

Though there are several papers presenting numerical solutions to the mixed-convection boundary-layer flow above a cooled horizontal plate none of these results is really satisfactory (Schneider, Steinrück & Andre 1994). All solutions agree near the edge of the plate but they differ significantly on where and how a singularity occurs. Schneider & Wasel (1985) were the first to find an unusual behaviour. They found a singularity with a finite wall shear stress. Later Wickern (1991*a, b*) claimed that the boundary-layer flow terminates in a Goldstein-type singularity. Daniels (1992) proved analytically the possibility of a singularity with an infinite wall shear stress. Considering that the numerical solution for the case of the boundary-layer flow above a heated horizontal or an inclined heated or cooled plate is straightforward the difficulties in the case of a cooled horizontal plate are surprising. In this paper we investigate the mathematical reason for this controversy.

The modified boundary-layer equations for the mixed-convection flow above a horizontal plate in dimensionless form are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}, \tag{1.1}$$

$$0=-\frac{\partial p}{\partial y}+\vartheta, \tag{1.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1.3}$$

$$u\frac{\partial \vartheta}{\partial x}+v\frac{\partial \vartheta}{\partial y}=\frac{1}{Pr}\frac{\partial^2 \vartheta}{\partial y^2}, \tag{1.4}$$

where the dimensionless coordinate x parallel to the plate is made dimensionless with the reference length $L = U_\infty^5/(g\beta\Delta T)^2\nu$ which depends on the velocity U_∞ of the free stream, the gravity acceleration g, the thermal expansivity β, the kinematic viscosity ν and the difference ΔT between a reference value of the plate temperature T_w and the temperature T_∞ of the undisturbed fluid. The dimensionless coordinate y perpendicular to the plate is scaled with $LRe^{-1/2}$, where $Re = U_\infty L/\nu$ is the Reynolds number. The velocity components u, v parallel and perpendicular to the plate are scaled with U_∞ and $U_\infty Re^{-1/2}$. The difference ϑ between the temperature of the disturbed and undisturbed fluid is scaled with ΔT. Reference values for the dimensionless skin friction τ and the dimensionless heat flux density q are $\rho_\infty U_\infty^2/Re$ and $k\Delta T Re^{-1/2}/L$, where ρ_∞ is the density of the undisturbed fluid and k is the thermal conductivity of the fluid. Note that according to this scaling we have $\tau = (\partial/\partial y)u(x, y = 0)$ and $q = (\partial/\partial y)\vartheta(x, y = 0)$.

In the classical boundary-layer equations the pressure p (which is scaled with $\rho_\infty U_\infty^2$) is determined by the outer flow and does not depend on the perpendicular coordinate y. Here, using Boussinesq's approximation, the boundary-layer equations are modified so that the hydrostatic pressure depends on buoyancy effects induced by the temperature difference ϑ with the unperturbed fluid. The Prandtl number $Pr = \nu/a$, with a the thermal diffusivity, is the only non-dimensional parameter in the problem.

The boundary conditions at the plate are given by the no-slip conditions and a prescribed temperature difference with the unperturbed fluid:

$$u(x,0) = 0, \quad v(x,0) = 0, \quad \vartheta(x,0) = \vartheta_w(x), \quad x > 0. \tag{1.5}$$

Since the solution of the boundary-layer equation has to match with the outer flow the asymptotic boundary conditions for $y \to \infty$ must hold:

$$u(x,\infty) = 1, \quad \vartheta(x,\infty) = 0, \quad p(x,\infty) = 0. \tag{1.6}$$

At the leading edge the flow is unperturbed, thus the initial conditions

$$u(0,y) = 1, \quad \vartheta(0,y) = 0, \quad y > 0, \tag{1.7}$$

hold. It is easy to verify that the modified boundary-layer equations (1.1)–(1.4) are of parabolic type in the sense defined by Courant & Hilbert (1968). Thus one might assume that (1.1)–(1.4) together with the boundary and initial conditions (1.5)–(1.7) is a well-posed problem. Indeed it is observed that (1.1)–(1.4) is well posed in case of a heated plate ($\vartheta_w > 0$). But this is not the case for a cooled plate.

A similarity solution exists for a plate temperature distribution of the form $\vartheta_w(x) = kx^{-1/2}$ for $k \geqslant k_0 < 0$ (Schneider 1979). In particular for $k \geqslant 0$ (heated plate) a unique similarity solution exists, while for $k_0 < k < 0$ (cooled plate) two similarity solutions exist. For $k < k_0$ no similarity solution exits at all. It turns out that the plate is adiabatic and that the heat transfer is concentrated at the leading edge of the plate. In this paper we will consider the case of a constant wall temperature $\vartheta_w = -1$ and point out the mathematical difficulties.

A necessary condition for the well posedness of a linear evolution problem

$$u_t = A(t)u, \quad u(0) = u_0, \tag{1.8}$$

is that the real part of the spectrum of the evolution operator A is bounded uniformly from above (Pazy 1983). This is the case for the heat equation or the wave equation on a bounded interval with Dirichlet boundary conditions and appropriate initial conditions. But it is not the case for the Laplace equation $u_{tt} = -u_{xx}$, with $u(-1,t) = u(1,t) = 0$, and the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

Since the boundary-layer equations are solved like an evolution system starting from the leading edge of the plate, we try to verify the above condition. Since (1.1)–(1.4) form a nonlinear system and the x-derivatives are not given explicitly we have to linearize (1.1)–(1.4) at a given solution and study locally the linearization which yields a generalized eigenvalue problem which will be analysed and the consequences of the results will be discussed.

2. Eigenvalues near the leading edge

To analyse the boundary-layer equations near the leading edge it is convenient to introduce the coordinate transform

$$\xi = (x/Pr)^{1/2}, \quad \eta = y/x^{1/2}. \tag{2.1}$$

Let $\psi(x,y)$ be a streamfunction; then we indroduce a transformed stream function f by

$$\psi(x,y) = x^{1/2} f(x^{1/2}/Pr^{1/2}, y/x^{1/2}). \tag{2.2}$$

The dependence of the coordinate transform (2.1) on the Prandtl number turns out to be useful when considering the limiting case of a small Prandtl number. From now on we will denote derivatives with respect to η with a prime and with respect to ξ with a subscript. The dimensionless skin friction τ and the dimensonless heat flux density q are given by

$$\tau = \frac{\partial^2}{\partial y^2}\psi = \frac{1}{Pr^{1/2}\xi} f'', \quad q = \frac{\partial}{\partial y}\vartheta = \frac{1}{Pr^{1/2}\xi}\vartheta'. \tag{2.3}$$

Thus (1.1)–(1.4) are equivalent to

$$2f''' + f f'' = \xi(f' f'_\xi - f'' f_\xi + g), \tag{2.4}$$

$$\frac{2}{Pr}\vartheta'' + f\vartheta' = \xi(f'\vartheta_\xi - \vartheta' f_\xi), \tag{2.5}$$

$$Pr^{-1/2} g' + \eta\vartheta' = \xi\vartheta_\xi, \tag{2.6}$$

with the boundary conditions

$$f(\xi,0) = f'(\xi,0) = \vartheta(\xi,0) + 1 = f'(\xi,\infty) - 1 = \vartheta(\xi,\infty) = g(\xi,\infty) = 0. \tag{2.7}$$

Note that the function g is the transformed pressure gradient parallel to the plate. Assuming a regular behaviour of f, ϑ and g near the leading edge we obtain ordinary differential equations for the initial values of f, ϑ and g where $f(0,\eta)$ satisfies the Blasius equation. We can construct formally a regular expansion of a solution of (2.4)–(2.7) by a power series expansion with respect to ξ (Afzal & Hussain 1984):

$$f_r^{(N)}(\xi,\eta) = \sum_{n=0}^{N} \xi^n f_n(\eta), \quad \vartheta_r^{(N)}(\xi,\eta) = \sum_{n=0}^{N} \xi^n \vartheta_n(\eta), \quad g_r^{(N)}(\xi,\eta) = \sum_{n=0}^{N} \xi^n g_n(\eta). \tag{2.8}$$

We denote by f_r, ϑ_r, g_r a solution of (2.4)–(2.7) with a regular expansion (2.8).

Let us assume we perturb a given solution at ξ_0 and let ΔF, $\Delta\vartheta$, ΔG denote the perturbation of f, ϑ, g. We are interested in whether the perturbation grows or decays locally. Thus we linearize (2.4)–(2.7), freeze the ξ-dependence of the coefficient functions and insert

$$\Delta F(\xi,\eta) = F(\eta)e^{\lambda\xi}, \quad \Delta\vartheta(\xi,\eta) = D(\eta)e^{\lambda\xi}, \quad \Delta G(\xi,\eta) = G(\eta)e^{\lambda\xi}, \tag{2.9}$$

into the linearized equations. This yields a generalized eigenvalue problem:

$$2F''' - \lambda\xi_0(f'F' - f''F) - \xi_0 G = -fF'' - f''F + \xi_0(f'_\xi F' - f_\xi F''), \tag{2.10}$$

$$\frac{2}{Pr}D'' - \lambda\xi_0(f'D - \vartheta'F) = -fD' - \vartheta'F + \xi_0(\vartheta_\xi F' - f_\xi D'), \tag{2.11}$$

$$Pr^{-1/2}G' - \lambda\xi_0 D = -\eta D', \tag{2.12}$$

$$F(0) = F'(0) = D(0) = F'(\infty) = D(\infty) = G(\infty) = 0. \tag{2.13}$$

Let us first consider the linearization near the leading edge of the plate. We solve the eigenvalue problem by an asymptotic expansion with respect to small values of ξ_0. We introduce the expansion

$$\lambda = \frac{\nu_0}{\xi_0} + \nu_1 + \nu_2\xi_0 + \ldots, \tag{2.14}$$

$$(F, D, G) = (F_0, D_0, G_0) + (F_1, D_1, G_1)\xi_0 + (F_2, D_2, G_2)\xi_0^2 + \ldots, \tag{2.15}$$

which yields the two decoupled eigenvalue problems for the leading terms of the asymptotic expansion:

$$2F_0''' + fF_0'' + f''F_0 = \nu_0(f'F_0' - f''F_0), \quad F_0(0) = F_0'(0) = F_0'(\infty) = 0, \tag{2.16}$$

$$\frac{2}{Pr}D_0'' + fD_0' + \vartheta'F_0 = \nu_0(f'D_0 - \vartheta'F_0), \quad D(0) = D(\infty) = 0. \tag{2.17}$$

Both eigenvalue problems (2.16), (2.17) can be written in self-adjoint form and therefore have only real eigenvalues. It was shown that (2.16) has only negative eigenvalues (Libby & Fox 1964). Multiplying (2.17) by D and integration by parts yields

$$\nu_0 \int_0^\infty f'D^2 \mathrm{d}\eta = -\int_0^\infty (2D'^2 + \frac{1}{2}f'D^2)\mathrm{d}\eta < 0. \tag{2.18}$$

Thus all eigenvalues which have an expansion of the form (2.14) are negative for ξ_0 sufficiently small. The same asymptotic expansion of the eigenvalues holds in the case of forced convection over a plate. Since all eigenvalues are negative this indicates that the evolution problem is well posed. However in the case of mixed convection a second type of asymptotic expansion of eigenvalues exists:

$$\lambda^+ = \frac{\lambda_0^+}{\xi_0^4} + \frac{\lambda_1^+}{\xi_0^3} + \ldots. \tag{2.19}$$

Since it turns out that this eigenvalue is positive we denote it and its corresponding eigenfunction with a superscript +. Inserting (2.19) into the eigenvalue problem yields a singularly perturbed eigenvalue problem and we have to expect a matched asymptotic expansion for the eigenfunction. For the outer expansion we use

$$F^+ = \bar{F}_0^+ + \xi_0\bar{F}_1^+ + \ldots \quad , \quad D^+ = \bar{D}_0^+ + \xi_0\bar{D}_1^+ + \ldots \quad , \quad G^+ = \frac{\bar{G}_0^+}{\xi_0^3} + \ldots \quad . \tag{2.20}$$

The leading-order terms of the asymptotic expansion satisfy

$$f'\bar{F}_0^{+'} - f''\bar{F}_0^+ = 0, \quad f'\bar{D}_0^+ - \vartheta'\bar{F}_0^+ = 0, \quad \bar{G}_0^{+'} = Pr^{1/2}\lambda_0^+\bar{D}_0^+, \tag{2.21}$$

with the solution

$$\bar{F}_0^+ = f', \quad \bar{D}_0^+ = \vartheta', \quad \bar{G}_0^+ = Pr^{1/2}\lambda_0^+\vartheta. \tag{2.22}$$

Obviously not all boundary conditions at $\eta = 0$ can be satisfied and an inner expansion is needed.

We define the inner expansion by

$$F^+ = \xi_0 \hat{F}_0^+\left(\frac{\eta}{\xi_0}\right), \quad D^+ = \hat{D}_0^+\left(\frac{\eta}{\xi_0}\right), \quad \zeta = \frac{\eta}{\xi_0}, \tag{2.23}$$

and obtain the equations for the inner expansion:

$$\hat{F}_0^{+'''} = a^3(\hat{F}_0^{+'} - \hat{F}_0^+), \quad \hat{D}_0^{+''} = b^3(\zeta\hat{D}_0^+ - \frac{\vartheta_0'(0)}{f_0''(0)}\hat{F}^+), \tag{2.24}$$

$a^3 = \lambda_0^+ f''(0)/2$ and $b^3 = Pr\, a^3$. We can express the solution in terms of the Airy function Ai:

$$\hat{F}_0^+ = \frac{f''(0)}{\int_0^\infty \mathrm{Ai}(as)\mathrm{d}s}\int_0^\zeta (\zeta - s)\mathrm{Ai}(as)\mathrm{d}s. \tag{2.25}$$

To determine λ_0^+ we have to consider the second-order equation. We obtain

$$f'\bar{F}_1^{+'} - f''\bar{F}_1^+ + Pr^{1/2}\vartheta_0 = 0, \tag{2.26}$$

with the solution

$$\bar{F}_1^+ = f'(\eta)\int_\infty^\eta \frac{\vartheta(s)}{f'^2(s)}\mathrm{d}s. \tag{2.27}$$

Matching the second-order outer expansion with the first-order inner expansion yields an equation for the eigenvalue λ_0^+:

$$\bar{F}_1^+(0) = \frac{Pr^{1/2}\vartheta(0)}{f''(0)} = \frac{f''(0)\mathrm{Ai}'(0)}{a\int_0^\infty \mathrm{Ai}(s)\mathrm{d}s} \tag{2.28}$$

and we finally obtain

$$\lambda_0^+ = 2f''(0)^5 Pr^{-3/2}\left(\frac{\mathrm{Ai}'(0)}{\vartheta(0)\int_0^\infty \mathrm{Ai}(s)\mathrm{d}s}\right)^3 = 0.00378 \times Pr^{-3/2} > 0. \tag{2.29}$$

3. Other large eigenvalues

Let us now look for large positive eigenvalues of (2.10)–(2.13) where ξ_0 is not small. For convenience we set

$$\varepsilon = (\lambda\xi_0)^{-1/3}. \tag{3.1}$$

We proceed similarly as in the previous section. We rescale $G = H/\varepsilon^3$ and expand the eigenfunction in terms of ε. For the first order of the outer expansion we obtain

$$f'^2\bar{F}'' - (f'f''' - Pr^{1/2}\xi_0\vartheta')\bar{F} = 0, \quad \bar{F}'(\infty) = 0. \tag{3.2}$$

Since f''' and ϑ decay for $\eta \to \infty$ the asymptotic behaviour of $\bar{F}$ for η large is given by

$$\bar{F} = d_1 + d_2\eta + o(1). \tag{3.3}$$

Thus the boundary condition at $\eta = \infty$ implies $d_2 = 0$ and we can choose $d_1 = 1$. Note that since $f'(0) = 0$ equation (3.2) is singular at $\eta = 0$. The behaviour of $\bar{F}$ for η small is given by

$$\bar{F}(\eta) = C_1 \eta^{\alpha_1} \bar{F}_1(\eta) + C_2 \eta^{\alpha_2} \bar{F}_2(\eta), \tag{3.4}$$

where α_1, α_2 are the solutions of the quadratic equation

$$\alpha^2 - \alpha + \frac{q}{\tau^2} = 0, \tag{3.5}$$

where τ and q are the dimensionless skin friction and heat flux density at the wall defined in (2.3) and $\bar{F}_1$, $\bar{F}_2$ are regular functions near $\eta = 0$ with $\bar{F}_i(0) = 1$. We remark that C_1 and C_2 are determined uniquely by $d_1 = 1$ and $d_2 = 0$.

For the inner expansion we use the expansion

$$F = \varepsilon^{\gamma} \hat{F}\left(\frac{\eta}{\varepsilon}\right), \quad D = \varepsilon^{\gamma-1} \hat{D}\left(\frac{\eta}{\varepsilon}\right), \quad G = \varepsilon^{\gamma-3} \hat{G}\left(\frac{\eta}{\varepsilon}\right), \quad \zeta = \frac{\eta}{\varepsilon}. \tag{3.6}$$

We obtain after some manipulations

$$2\hat{F}^{(\mathrm{vi})} - Pr^{1/2}\xi_0(1 + Pr)\tau\zeta\hat{F}^{(\mathrm{iv})} + 2Pr^{1/2}\xi_0\tau\hat{F}''' + \frac{1}{2}Pr^2\xi_0^2(\tau^2\zeta^2\hat{F}'' - q\hat{F}) = 0, \tag{3.7}$$

$$\hat{F}(0) = \hat{F}'(0) = \hat{F}^{(\mathrm{iv})}(0) = 0. \tag{3.8}$$

The equations for the inner expansion have two independent solutions $\hat{F}_3, \hat{F}_4$ which decay exponentially , two independent solutions $\hat{F}_5, \hat{F}_6$ which increase faster than exponentially and two solutions $\hat{F}_1 \sim \zeta^{\alpha_1}$, $\hat{F}_2 \sim \zeta^{\alpha_2}$ which behave algebraically as ζ tends to infinity. The exponents α_i are again the solutions of the quadratic equation (3.5).

Therefore a solution of (3.7) has to be a linear combination of F_1, F_2, F_3, F_4 and for $\zeta \to \infty$ the asymptotic behaviour of $\hat{F}$ is given by

$$\hat{F}(\zeta) \sim c_1 \zeta^{\alpha_1} + c_2 \zeta^{\alpha_2}. \tag{3.9}$$

We suppose that the ratio c_1/c_2 can be determined uniquely by the boundary conditions (3.8) at $\zeta = 0$.

Matching the inner and outer solutions we have to discuss several cases as follows.

3.1. α_j *real*

This is the case if $\tau^2 \geqslant 4q$. We assume $\alpha_1 \leqslant \alpha_2$. Then we have to distinguish two subcases:

(i) $c_2 \neq 0$, for which we have $c_2 = C_2$, and $\gamma = \alpha_2$;

(ii) $c_2 = 0$, then we have $c_1 = C_1$, and $\gamma = \alpha_1$.

In the first case the non-viscous bulk flow is reponsible for the large positive eigenvalue. Matching the inner and outer solutions yields $C_1 = \varepsilon^{\alpha_2 - \alpha_1} c_1$ and thus we can supplement (3.2) by the boundary condition

$$\lim_{\eta \to 0} \eta^{-\alpha_1} \bar{F}(\eta) = 0, \tag{3.10}$$

to obtain an eigenvalue equation. Note that this problem is similar to the eigenvalue problem in Daniels (1992) for the occurrence of a singularity in a thermal boundary-layer flow.

In the second case the viscous sublayer is responsible for the large eigenvalue. We can supplement (3.7) with the asymptotic boundary condition

$$\lim_{\zeta \to \infty} \zeta^{-\alpha_2} \hat{F}(\zeta) = 0. \tag{3.11}$$

Eigenfunctions and eigenvalues of the eigenvalue problem (3.2), (3.10) and (3.7), (3.8), (3.11) have apparently not been observed numerically.

3.2. α_j *complex*

In this case we write $\alpha_i = 1/2 \pm \mathrm{i}\delta$ with

$$\delta = \left(\frac{q}{\tau^2} - \frac{1}{4}\right)^{1/2}, \tag{3.12}$$

and have $\gamma = 1/2$. Thus the outer solution for small values of η is given by

$$\bar{F}(\eta) = \eta^{1/2}(C_1 \cos(\delta \log \eta)(1 + o(1)) + C_2 \sin(\delta \log \eta)(1 + o(1)). \tag{3.13}$$

For the inner solution we have the asymptotic expansion for large values of ζ:

$$\hat{F}(\zeta) = \zeta^{1/2}(C_1 \cos(\delta \log \zeta) + C_2 \sin(\delta \log \eta)). \tag{3.14}$$

Assuming that the boundary conditions at $\zeta = 0$ for the inner expansion determine the ratio c_1/c_2 uniquely and that the asymptotic boundary condition at $\eta = \infty$ for the outer expansion determine the ratio C_1/C_2 uniquely the matching condition yields

$$\frac{c_1}{c_2} = \frac{(C_1/C_2) + \tan \delta \log \varepsilon}{1 - (C_1/C_2) \tan \delta \log \varepsilon}, \tag{3.15}$$

an asymptotic equation for large positive eigenvalues. Note that (3.15) has infinitely many solutions. Since it holds only asymptotically only those solutions with ε sufficiently small are of interest. The corresponding eigenvalues are given by

$$\lambda_n \sim \lambda_0 \exp \frac{n\pi}{\delta}, \quad \text{with} \quad n \in N_0, \tag{3.16}$$

where λ_0 is a sufficiently large positive eigenvalue. In other words, if the skin friction is smaller than a certain value depending only on the heat flux density $\tau^2 < 4q$ an unbounded sequence of positive eigenvalues exists. In §6 we present numerical results showing the first five positive eigenvalues as functions of ξ along a solution branch (figure 3) and the corresponding F'-component of the corresponding eigenfunctions for $\xi = 0.101$ (figure 4).

Since a well-posed linear evolution problem must have spectrum with its real part bounded from above we cannot treat the boundary-layer equation as an evolution problem any more.

4. Non-uniqueness of the solution

Now let us discuss a consequence of the existence of the positive eigenvalue $\lambda^+ = \lambda_0^+/\xi^4(1 + O(\xi))$ of the linearized problem. Besides the regular expansion (2.8) another type of expansion of a solution exists. To motivate the type of expansion we consider a simple initial value problem whose evolution operator has an eigenvalue with an expansion like λ_+. The general solution of the ordinary differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}\xi} = \frac{\lambda_0^+}{\xi^4} z, \quad z(0) = 0 \tag{4.1}$$

is given by the one-parametric family

$$z(\xi) = c \exp\left(-\frac{\lambda_0^+}{3\xi^3}\right) \tag{4.2}$$

with an arbitrary constant c. A similar situation holds in the case of the modified boundary-layer equations.

Let f_r, ϑ_r, g_r be a solution of (2.4)–(2.7) with the regular expansion (2.8). Then using the expansion of the eigenvalue λ^+ and of the corresponding eigenfunction we can construct an asymptotic expansion of an alternative solution:

$$f = f_r(\xi,\eta) + \sum_{k\geq 1} c^k \exp\left(-\frac{k\Lambda(\xi)}{3\xi^3}\right)\tilde{f}_k(\xi,\eta), \tag{4.3}$$

$$\vartheta = \vartheta_r(\xi,\eta) + \sum_{k\geq 1} c^k \exp\left(-\frac{k\Lambda(\xi)}{3\xi^3}\right)\tilde{\vartheta}_k(\xi,\eta), \tag{4.4}$$

$$g = g_r(\xi,\eta) + \sum_{k\geq 1} c^k \exp\left(-\frac{k\Lambda(\xi)}{3\xi^3}\right)\tilde{g}_k(\xi,\eta), \tag{4.5}$$

where c is an arbitrary constant and Λ, $\tilde{f}_k$, $\tilde{\vartheta}_k$, $\tilde{g}_k$ have asymptotic expansions of the form

$$\tilde{f}_k = \begin{cases} \sum_{l\geq 0} \xi^l \bar{f}_{kl}(\eta) & \text{(outer expansion)} \\ \sum_{l\geq 1} \xi^l \hat{f}_{kl}(\xi/\eta) & \text{(inner expansion),} \end{cases} \tag{4.6}$$

$$\tilde{t}_k = \begin{cases} \sum_{l\geq 0} \xi^l \bar{\vartheta}_{kl}(\eta) \\ \sum_{l\geq 0} \xi^l \hat{\vartheta}_{kl}(\xi/\eta) \end{cases}, \quad \tilde{g}_k = \begin{cases} \sum_{l\geq -3} \xi^l \bar{g}_{kl}(\eta) \\ \sum_{l\geq -3} \xi^l \hat{g}_{kl}(\xi/\eta) \end{cases} \tag{4.7}$$

and

$$\Lambda(\xi) = \sum_{l\geq 1} \xi^{l-1}\Lambda_l. \tag{4.8}$$

Inserting (4.3)–(4.5) and comparing equal powers of $\exp(-\Lambda/\xi^3)$ yields

$$\begin{aligned} &2\tilde{f}_k''' + f_r\tilde{f}_k'' + \tilde{f}_k f_r'' - \xi\tilde{g}_k \\ &\quad -\xi f_r'\left(\tilde{f}_{k,\xi}' + \frac{\Lambda k}{\xi^4}\tilde{f}_k' - \frac{\Lambda' k}{3\xi^3}\tilde{f}k'\right) - \xi\tilde{f}_k' f_{r,\xi}' + \xi f_r''\left(\tilde{f}_{k,\xi} + \frac{\Lambda k}{\xi^4}\tilde{f}_k - \frac{\Lambda' k}{3\xi^3}\tilde{f}_{k,\xi}\right) + \xi\tilde{f}_k'' f_{r,\xi} \\ &= \sum_{m=1}^{k-1}\left(-\tilde{f}_m\tilde{f}_{k-m}'' + \xi\tilde{f}_{k-m}'\left[\tilde{f}_{m,\xi}' + \frac{\Lambda m}{\xi^4}\tilde{f}_m' - \frac{\Lambda' m}{3\xi^3}\tilde{f}_m'\right]\right. \\ &\qquad\qquad \left. -\xi\tilde{f}_{k-m}''\left[\tilde{f}_{m,\xi} + \frac{\Lambda m}{\xi^4}\tilde{f}_m - \frac{\Lambda' m}{3\xi^3}\tilde{f}_m\right]\right). \end{aligned} \tag{4.9}$$

The equations for $\tilde{\vartheta}_k$ and $\tilde{g}_k$ are of a similar structure. Now inserting the expansions for f_r, ϑ_r, g_r and (4.6)–(4.8) and comparing equal powers of ξ we obtain equations for $\bar{f}_{kl}$, $\bar{\vartheta}_{kl}$, $\bar{g}_{kl}$, $\hat{f}_{kl}$, $\hat{\vartheta}_{kl}$, $\hat{g}_{kl}$ and Λ_l. The equations for $\bar{f}_{10}$, $\hat{f}_{10}$, $\bar{\vartheta}_{10}$, $\hat{\vartheta}_{10}$, $\bar{g}_{10}$, $\bar{f}_{11}$ and Λ_1 are exactly the same as for $\bar{F}_0^+$, $\bar{D}_0^+$ $\bar{G}_0^+$, $\hat{F}_0^+$ $\bar{F}_1^+$ and λ_0^+ and thus the leading-order terms of the expansion of the alternative solution are just the leading-order terms of the eigenfunction (F^+, D^+, G^+) and $\Lambda_1 = \lambda_0^+$.

Note that in order to determine the klth term of the expansion we need only terms with index jm where $0 \leqslant j \leqslant k$, and $0 \leqslant m \leqslant l$. The alternative solution (4.3)–(4.5) form a one-parametric family of solutions parametrized by c. The expansion is only valid on an interval $(0, \xi_0)$ where $c \exp -A_1/3\xi_0^3 \ll 1$.

Starting with $\xi = 0$ the alternative solutions are at first very close to the solution with the regular expansion, but near $\xi_c \sim (A_1/3\ln(c))^{1/3}$ they branch off very rapidly.

5. Propagation of perturbations in a numerical scheme

In this section we will investigate how a small perturbation is propagated in a discrete scheme for the numerical solution of the modified boundary-layer equations. For simplicity we will study only the implicit Euler scheme for the discretization in the ξ-direction and assume the η-direction is not discretized. Let $f_k(\eta)$, $\vartheta_k(\eta)$, $g_k(\eta)$ be approximations to f, ϑ, g at the mesh points $\xi_k = k\Delta\xi$. Then the implicit Euler scheme is given by

$$2f_k''' + f_k f_k'' = \xi_k \left(f_k' \frac{f_k' - f_{k-1}'}{\Delta\xi} - f_k'' \frac{f_k - f_{k-1}}{\Delta\xi} + g_k \right), \tag{5.1}$$

$$\frac{2}{Pr}\vartheta_k'' + f_k \vartheta_k' = \xi_k \left(f_k' \frac{\vartheta_k - \vartheta_{k-1}}{\Delta\xi} - \vartheta_k' \frac{f_k - f_{k-1}}{\Delta\xi} \right), \tag{5.2}$$

$$Pr^{-1/2} g' + \eta \vartheta' = \xi_k \frac{\vartheta_k - \vartheta_{k-1}}{\Delta\xi}, \tag{5.3}$$

$$f_k(\xi,0) = f_k'(\xi,0) = \vartheta_k(\xi,0) + 1 = f_k'(\xi,\infty) - 1 = \vartheta_k(\xi,\infty) = g_k(\xi,\infty) = 0. \tag{5.4}$$

To study how a perturbation propagates we assume that f_{k-1}, f_k, ϑ_{k-1}, ϑ_k, g_{k-1}, g_k is a given numerical solution, linearize (5.1)–(5.4) and make an eigenvalue ansatz

$$F_k = \mu F_{k-1}, \quad \vartheta_k = \mu \vartheta_{k-1} \quad g_k = \mu g_{k-1}, \tag{5.5}$$

which yields the eigenvalue problem

$$2F_k''' - \frac{\mu - 1}{\mu\Delta\xi} \xi_k (f_k' F_k' - f_k'' F_k) - \xi_k G_k = -f_k F_k'' - f_k'' F_k + \xi_k \left(\frac{f_k' - f_{k-1}'}{\Delta\xi} F_k' - \frac{f_k - f_{k-1}}{\Delta\xi} F_k'' \right), \tag{5.6}$$

$$\frac{2}{Pr} D_k'' - \frac{\mu - 1}{\mu\Delta\xi} \xi_k (f_k' D_k - \vartheta_k' F_k) = -f D' - \vartheta' F + \xi_k \left(\frac{\vartheta_k - \vartheta_{k-1}}{\Delta\xi} F_k' - \frac{f_k - f_{k-1}}{\Delta\xi} D_k' \right), \tag{5.7}$$

$$Pr^{-1/2} G_k' - \frac{1 - \mu}{\mu\Delta\xi} \xi_k D_k = -\eta D_k', \tag{5.8}$$

$$F_k(0) = F_k'(0) = D_k(0) = F_k'(\infty) = D_k(\infty) = G_k(\infty) = 0. \tag{5.9}$$

Setting $\tilde{\lambda} = (\mu - 1)/(\mu\Delta\xi)$ we obtain the eigenvalue problem (2.10)–(2.13) where the derivatives with respect to ξ are replaced by difference quotients. Assuming that the given numerical solution approximates a smooth solution and that λ is a simple eigenvalue we have

$$\tilde{\lambda} = \lambda(1 + O(\Delta\xi)), \tag{5.10}$$

and the corresponding propagation factor is

$$\mu = \frac{1}{1 - \lambda\Delta\xi(1 + O(\Delta\xi))}. \tag{5.11}$$

A perturbation in the direction of the eigenvector corresponding to the eigenvalue λ is damped if $|\mu| < 1$. This is the case if $\lambda < 0$ or $\lambda > 2/\Delta\xi$. In the first case the numerical scheme agrees with the differential equation; in the second case the numerical scheme damps the perturbation although the differential equation amplifies it. Obviously the stepsize is too large to resolve the differential equation correctly. For $\lambda\Delta\xi \sim 1$ the propagation factor tends to infinity and the numerical method collapses. Note that if $\lambda\Delta\xi = 1$ the linearization of the implicit Euler scheme is singular. To avoid these difficulties one has to choose the stepsize $\Delta\xi$ sufficiently small. This is only possible if the real part of the eigenvalues is bounded uniformly from above.

Now let us discuss what happens when the modified boundary-layer equations are solved by using the implicit Euler scheme. We have a large positive eigenvalue near the leading edge of the plate with an asymptotic expansion of the form (2.19). All perturbations are damped for

$$\xi < (\tfrac{1}{2}\lambda_0^+\Delta\xi)^{1/4} \tag{5.12}$$

and the numerical solution seems acceptable. But in the interval

$$(\tfrac{1}{2}\lambda_0^+\Delta\xi)^{1/4} < \xi < (\lambda_0^+\Delta\xi)^{1/4} \tag{5.13}$$

oscillations with an increasing amplitude will occur and near

$$\xi_{crit} = C_{crit}\xi^{1/4}, \quad \text{with} \quad C_{crit} = (\lambda_0^+)^{1/4} = 0.247 \times Pr^{-3/8}, \tag{5.14}$$

the numerical solution terminates in a singularity.

We remark that the arguments presented here for the implicit Euler scheme can be carried over to any implicit scheme. For the Keller box scheme Wickern (1987) found the relation (5.14) but with a different constant $C_{crit} = 0.235 \times Pr^{-3/8}$ by numerical experiments.

6. Numerical results

In the previous section we have shown that the initial value problem (2.4)–(2.7) has no unique solution. A one-parametric family of solutions with a singular expansion branches off from a solution with a unique regular expansion. However the regular expansion is valid only for ξ sufficiently small. Therefore we want to follow this solution numerically. Using an implicit numerical method we will run into difficultis at ξ_{crit} which depends on the stepsize $\Delta\xi$. Therefore we choose first a stepsize and then pose the initial condition using the regular expansion at a point $\xi_0 > \xi_{crit}(\Delta\xi)$. Since the solution is very sensitive to a perturbation of the initial condition in the direction of the eigenfunction F_+, D_+ corresponding to the positive eigenvalue λ_+ we perturb the initial condition:

$$f(\xi_0,\eta) = f_r^{(n)} + \sigma F^+(\eta), \quad \vartheta(\xi_0,\eta) = \vartheta_r^{(n)} + \sigma D^+(\eta). \tag{6.1}$$

We have normalized F^+ by $F^{+\prime\prime}(0) = 1$. The constant σ is arbitrary and we will solve the initial value problem for several values of σ.

We solve the equations (2.4)–(2.7) for $Pr = 0.72$ (air) using the implicit Euler scheme with a stepsize $\Delta\xi = 10^{-4}$. In figure 1 we have plotted the skin friction coefficient $f''(0)$ and in figure 2 we have plotted the heat transfer at the plate $\vartheta'(0)$ as functions of ξ for different initial conditions.

We start at $\xi_0 = 0.05$ from the regular expansion with $n = 8$. For $\sigma = -0.001465$, -0.001450, -0.001442 the skin friction decreases very fast and finally vanishes. Note

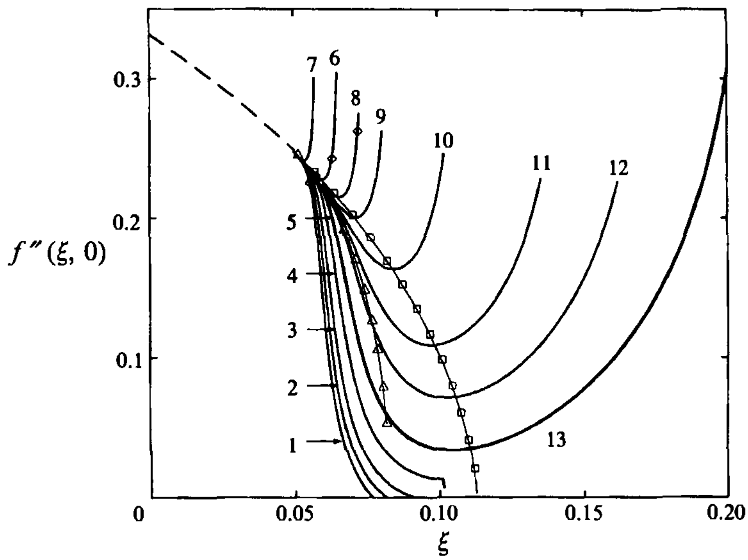

FIGURE 1. Wall shear stress $f''(\xi,0) = \tau/Pr^{1/2}\xi$ for $Pr = 0.72$. Regular expansion with 8 terms: dashed line. Numbered curves show solution of inital value problem with initial data $f(\xi_0,\eta) = f_r^{(8)}(\xi_0,\eta) + \sigma F_+(\eta)$, $\vartheta(\xi_0,\eta) = \vartheta_r^{(8)}(\xi_0,\eta) + \sigma D_+(\eta)$ for various values of ξ_0 and σ (see table 1). ◇, $\delta^2 = \frac{1}{4} - q/\tau^2 = 0$; △ – △, Schneider & Wasel (1985); □ – □, Wickern (1991*a, b*).

$\xi_0 = 0.05$		$\xi_0 = 0.06$	
label	σ	label	σ
1	-0.001465	8	-0.003906
2	-0.001450	9	-0.004395
3	-0.001442	10	-0.004639
4	-0.001438	11	-0.004883
5	-0.001437	12	-0.005371
6	-0.001436	13	-0.007813
7	-0.001404		

TABLE 1. Initial data

that there is no singularity of the Goldstein type. For $\sigma = -0.001404, -0.001436$ the skin friction increases very fast and becomes singular. It looks as if the different solutions branch off from the regular expansion which is in accordance with the non-uniqueness result of §4. We use $\xi_0 = 0.06$ as a second starting point. For $\sigma = -0.003906$ the solution first seems to follow the 'regular' solution and then increases very fast. Increasing σ shows that all solutions first decay then increase and finally tend to infinity. However it is not possibly to see the 'regular' solution from where the increasing and decaying solutions branch off. Any solution with σ between -0.0044395 and -0.005371 can be a continuation of the 'regular' solution.

The solution with starting point $\xi_0 = 0.05$ and $\sigma = -0.001437$ seems to be identical with the solution with the starting point $\xi_0 = 0.06$ and $\sigma = -0.007813$: it first decays very rapidly, then reaches a minimum, increases slowly and finally tends to infinity.

Another interesting solution is obtained for the starting point $\xi = 0.05$ and $\sigma = -0.001438$. First the skin friction decreases, then it reaches almost a positive

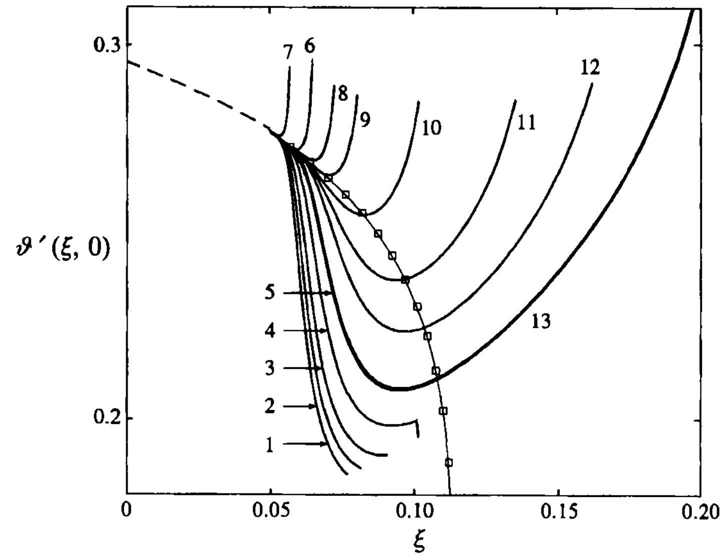

FIGURE 2. Heat transfer at the plate $\vartheta'(\xi,0) = q/Pr^{1/2}\xi$ for $Pr = 0.72$. Regular expansion with 8 terms: dashed line. Numbered curves show solution of inital value problem with initial data $f(\xi_0,\eta) = f_r^{(8)}(\xi_0,\eta)+\sigma F_+(\eta)$, $\vartheta(\xi_0,\eta) = \vartheta_r^{(8)}(\xi_0,\eta)+\sigma D_+(\eta)$ for various values of ξ_0 and σ (see table 1). □ – □, Wickern (1991*a,b*).

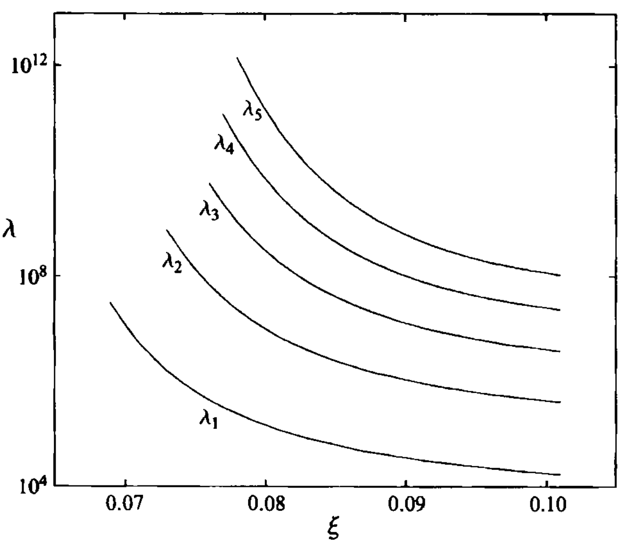

FIGURE 3. Positive eigenvalues $\lambda_1 \cdots \lambda_5$ of (2.10)–(2.13) along the solution of the initial value problem with inital data $\xi_0 = 0.05$, $\sigma = -0.001438$ as functions of ξ.

minimum and suddenly this solution branch terminates in a numerical singularity. In order to verify that the numerical singularity is due to a large positive eigenvalue such as described in §3 we compute the eigenvalues and eigenfunctions along this solution. In figure 3 we have plotted the first five positive eigenvalues as functions of ξ. Since these eigenvalues only exist for $\tau < 2q^{1/2}$ we have marked the points where

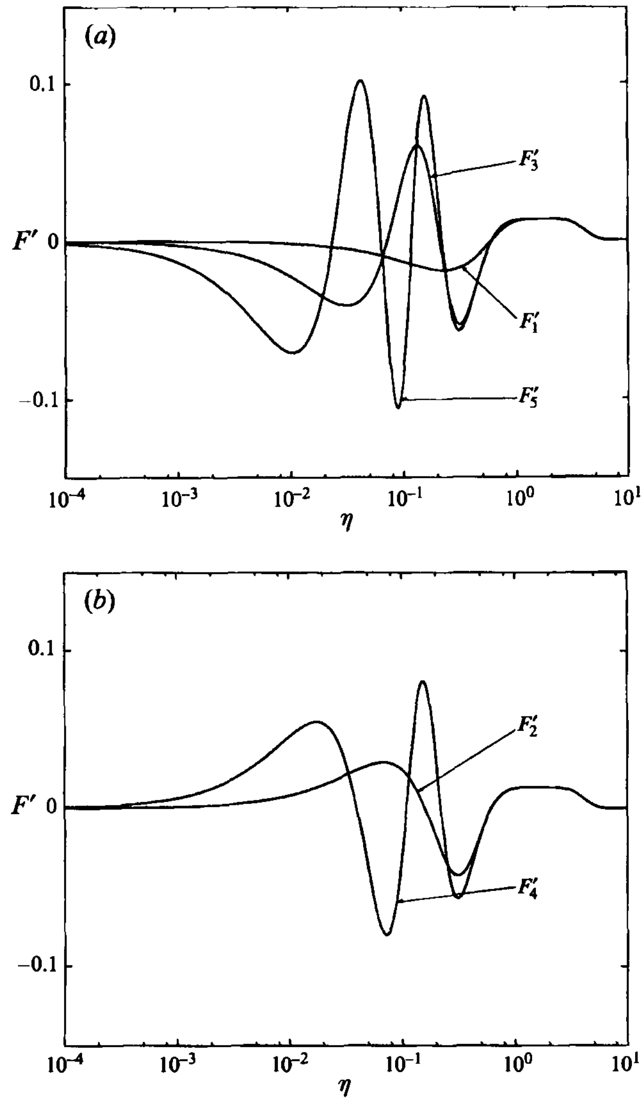

FIGURE 4. (*a*) Eigenfunctions $F_1'(\eta)$, $F_3'(\eta)$, $F_5'(\eta)$ corresponding to the eigenvalues $\lambda_1 = 1.7 \times 10^4$, $\lambda_3 = 3.9 \times 10^6$, $\lambda_5 = 1.1 \times 10^8$ of (2.10)–(2.13), and (*b*) eigenfunctions $F_2'(\eta)$, $F_4'(\eta)$ corresponding to the eigenvalues $\lambda_1 = 4.2 \times 10^5$, $\lambda_3 = 2.3 \times 10^7$ of (2.10)–(2.13). (f, ϑ, g) is the solution of the initial value problem with initial data $\xi_0 = 0.05, \sigma = -0.001438$ at $\xi = 0.101$.

$\tau = 2q^{1/2}$ holds with ◇ in figure 1. We observe that near the numerical singularity the smallest of these eigenvalues is close to $1/\Delta\xi = 10^4$.

In figure 4(*a*,*b*) we have plotted the F' component of the corresponding eigenfunctions at $\xi = 0.101$. As predicted by the asymptotic analysis these eigenfunctions are almost identicical for $\eta > 0.5$.

In figure 5 we have plotted $\lambda^{1/3}F'/F''(0)$ as a function of the local variable $\eta\lambda^{1/3}$. Again we observe that in that scaling the eigenfunctions agree near the wall as predicted by the asymptotic analysis. The eigenfunctions differ only in the number of

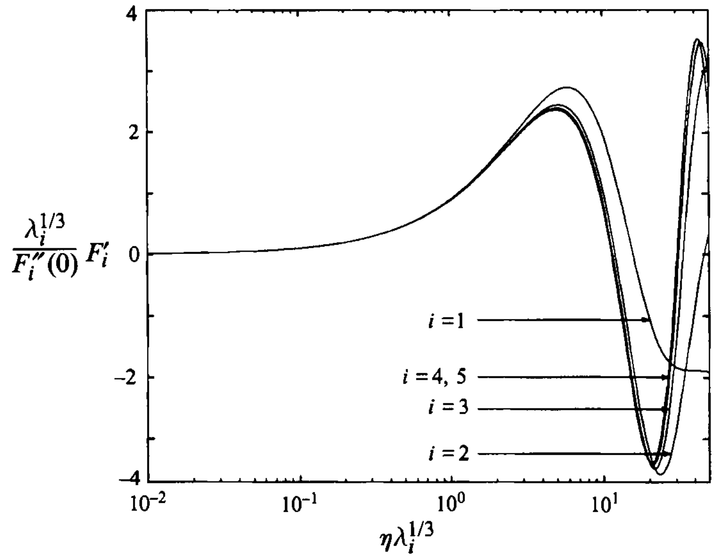

FIGURE 5. Eigenfunctions $F_1', \ldots, F_5'$ corresponding to the eigenvalues $\lambda_1 \ldots \lambda_5$ as functions of the inner variable $\zeta = \eta\lambda^{1/3}$ of (2.10)–(2.13). (f, ϑ, g) is the solution of the initial value problem with initial data $\xi_0 = 0.05, \sigma = -0.001438$ at $\xi = 0.101$.

extreme values in the intermediate zone beetween the outer variable η and the inner variable $\zeta = \eta/\lambda^{1/3}$.

Wickern (1987) investigated the numerical solution of the modified boundary-layer equations most thoroughly of all previous authors. He compared two different discretization methods: a 'lagging' method used by Schneider & Wasel (1985) and a modified Keller box scheme. In both cases the numerical solution terminated in a singularity which depends on the stepsize in the ξ-direction. For the case of the Keller box scheme he found empirically $\xi_{crit} \sim \Delta\xi^{1/4}$.

To avoid the numerical difficulties Wickern (1991*a,b*) used the regular expansion (2.8) to prescribe parts of the pressure gradient: equation (2.6) is replaced by

$$Pr^{-1/2}g' + \eta\vartheta' + \vartheta = \xi\frac{\partial}{\partial\xi}\vartheta_r^{(4)} + \vartheta_r^{(4)}, \tag{6.2}$$

where $\vartheta_r^{(4)}$ are the first four terms of the regular expansion (2.8). With this ad hoc assumption a singularity of Goldstein type is obtained, which seems to be incorrect. For comparison we have plotted the results for the skin friction of Schneider & Wasel (1985) and Wickern (1991*a,b*) in figure 1.

7. Conclusions

We have shown by an asymptotic analysis that the real part of the spectrum of the linearization is not bounded uniformly from above which explains the numerical difficulties observed by previous authors (Schneider & Wasel 1985; Wang & Kleinstreuer 1990; Wickern 1987). Furthermore we proved a relation between the stepsize in the ξ-direction and the location of the numerical singularity which was found empirically by Wickern (1987).

Two types of large positive eigenvalues exist. Near the leading edge of the plate a large positive eigenvalue exists and secondly if the wall shear stress is less than a critical value depending only on the heat flux on the plate an unbounded sequence of positive eigenvalues exists. Both types of eigenvalues are found by an asymptotic expansion.

Since all computed solutions branch off from a very small neighbourhood of the regular expansion and since we have shown that the initial value problem starting at $\xi = 0$ has no unique solution we conclude that all computed solutions may correspond to solutions of the boundary-layer equations. But we cannot choose any of these solution as the physical relevant solution.

The behaviour near the point where $\tau^2 = 4q$ has not been discussed yet. Since the differential equation is singular at this point due to the large positive eigenvalues, we expect that solutions can branch off from a given solution. This behaviour is reflected in the numerical singularity which occurs when $\lambda\Delta\xi \sim 1$ holds.

Further research is needed into which of these boundary-layer solutions is of physical relevance and can approximate a solution of the original Navier–Stokes equations.

I want to thank Professor W. Schneider for drawing my attention to this boundary-layer flow problem and for many discussions about the physical consequences of these formal properties of the modified boundary-layer equations. The support by the Austrian Science Foundation (FWF), Project No. P9584-TEC is also acknowledged.

REFERENCES

AFZAL N. & HUSSAIN, T. 1984 Mixed convection over a horizontal plate. *Trans. ASME* C: *J. Heat Transfer* **106**, 240–241.

CHEN, T. S., SPARROW, E. M. & MOCUGLU, A. 1977 Mixed convection in boundary layer flow on a horizontal plate. *Trans. ASME* C: *J. Heat Transfer* **99**, 66–71.

COURANT, R. & HILBERT, D. 1968 *Methoden der Mathematischen Physik II*, 2 Aufl. Springer.

DANIELS, P. G. 1992 A singularity in thermal boundary-layer flow on a horizontal surface. *J. Fluid Mech.* **242**, 419–440.

LIBBY, P. & FOX, H. 1964 Some perturbation solutions in laminar boundary layer theory. Part 1. The momentum equation. *J. Fluid Mech.* **17**, 433–449.

PAZY, A. 1983 *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer.

SCHNEIDER, W. 1979 A similarity solution for combined forced and free convection flow over a horizontal plate. *Intl J. Heat Transfer* **22**, 1401–1406.

SCHNEIDER, W., STEINRÜCK, H. & ANDRE, G. 1994 The breakdown of boundary layer computations in case of the flow over a cooled horizontal flat plate. *Z. Angew. Math. Mech.* **74**, T402–T404.

SCHNEIDER, W. & WASEL, M. G. 1985 Breakdown of the boundary-layer approximation for mixed convection above a horizontal plate. *Intl J. Heat Mass Transfer* **28**, 2307–2313.

WANG, T. & KLEINSTREUER, C. 1990 Boundary layer analysis of orthogonal free-forced convection on a heated and a cooled plate with fluid injection or suction. *Intl J. Engng Sci.* **28**, 5, 437–450.

WICKERN, G. 1987 Untersuchung der laminaren gemischten Konvektion an einer beliebig geneigten ebenen Platte mit besonderer Berücksichtigung der Strömungsablösung. Thesis, Ruhr-Univ. Bochum, Fortschr.-Ber. VDI 7 Nr. 129.

WICKERN, G. 1991*a* Mixed convection from an arbitrarily inclined semi-infinite flat plate – I. The influence of the inclination angle. *Intl J. Heat Mass Transfer* **34**, 1935–1945.

WICKERN, G. 1991*b* Mixed convection from an arbitrarily inclined semi-infinite flat plate – II. The influence of the Prandtl number. *Intl J. Heat Mass Transfer* **34**, 1947–1957.